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Singularity analysis for the mean curvature flow

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Abstract: We consider the geometry of the first time singularities of the mean curvature flow of closed, smooth hypersurfaces in the Euclidean space $\mathbf{R}^{n+1}$. A crucial estimate by Huisken and Sinestrari is the curvature pinching estimate for mean convex flow, that is mean curvature flow of hypersurfaces with positive mean curvature,

$$\kappa_1 \geq -\varphi(\kappa_1 + \cdots + \kappa_n),$$

where $\kappa_1 \leq \cdots \leq \kappa_n$ are the principal curvatures and φ is a nonnegative function satisfying $\varphi(t) \to 0$ as $t \to \infty$. This estimate corresponds to the well know curvature pinching estimate of Hamilton and Ivey for the Ricci flow of 3-manifolds.

By the curvature pinching estimate and some basic estimates for uniformly parabolic equations, we have the following results for the first time singularities.

Theorem 1. Let $\mathcal{F} = \{F_t : \ t \in [0, T)\}$ be a mean convex flow of closed hypersurface in $\mathbf{R}^{n+1}$. Then up to the first time singularities,
(i) Any blow-up sequence $\mathcal{F}^k = \{F_t^k\}$ converges locally smoothly along a subsequence to a blow-up solution $\mathcal{F}' = \{F_t'\}_{t\in(-\infty,T')}$, where $T' \geq 0$.
(ii) For any $t < T'$, F_t' is a convex hypersurface with nonempty interior.
(iii) The grim reaper and any pair of parallel hyperplanes are not blow-up solutions.
(iv) The set of blow-up solutions of normalized blow-up sequences is compact.

The above theorem was proved by White using the geometric measure theory and is based on earlier work of Brakke and Ilmanen. We will present a new proof using the curvature pinching estimate.

We also consider the classification of blow-up solutions to the mean curvature flow.

Theorem 2.

(i) A blow-up translating solution in $\mathbf{R}^3$ is rotationally symmetric.

(ii)There exists non-rotationally symmetric translating solution in $\mathbf{R}^{n+1}$ for $n > 2$.

(iii) The blow-down of a blow-up solution is a shrinking sphere or cylinder.

(iv) Let $\mathcal{F} = \{F_t\}$ be a mean convex flow in $\mathbf{R}^3$. Then at any point $p = (x, t)$ with sufficiently large mean curvature, F_t is ε-close to a canonical blow-up solution.

The above two theorems show that the singularity profile of the mean convex flow is in line with that of Ricci flow of 3-manifolds obtained by Hamilton and Perelman. Similar results for 2-convex (i.e. $\kappa_1 + \kappa_2 > 0$) mean curvature flow with surgery have also been obtained by Huisken and Sinestrari.